HOMEWORK 1

Problem 1 Prove that there are infinitely many prime numbers of the form

(a) $4n+3$;

(b) $6n+5$.

Problem 2 Prove that for every natural N there exist infinitely many n such that the sequence $n, n+1, \dots, n+N-1$ contains no prime numbers. (*Hint*: construct such n for a given N).

Problem 3 Prove that the following numbers are not integers:

(a) $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}, \quad n>1$;

(b) $\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}, \quad n>0$.

Problem 4 For real x denote by $[x]$ the integer part function — largest integer which does not exceed x.

(a) Let $f(x)$ be continuous non-negative function in the interval $0 \le x \le R$. Prove that the number of integral points (points whose coordinates are integers) in the region $0 < x \le R, \;\; 0 < y \le f(x)$ is given by

$$\sum_{Q<x\le R} [f(x)]$$

(summation goes over integer x).

(b) If P and Q are odd and $(P,Q)=1$, prove that

$$\sum_{0<x<\frac{Q}{2}} \left[\frac{P}{Q}x\right] + \sum_{0<y<\frac{P}{2}} \left[\frac{Q}{P}y\right] = \frac{P-1}{2}\cdot\frac{Q-1}{2}.$$

(c) Prove that the number of integral points inside the circle $x^2+y^2=r^2$, $r>0$, is given by

$$1+4[r]+8\sum_{0<x\le\frac{r}{\sqrt{2}}}\left[\sqrt{r^2-x^2}\right]-4\left[\frac{r}{\sqrt{2}}\right]^2.$$

(d) Prove that the number of integral points in the domain $x>0, y>0$ and $0<xy\le n$ is given by

$$2\sum_{0<x\le\sqrt{n}}\left[\frac{n}{x}\right]-\left[\sqrt{n}\right]^2.$$